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Mayer Coefficients in Two-Dimensional Coulomb Systems

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We show that, for neutral systems of particles of arbitrary charges in two dimensions, with hard cores, coefficients of the Mayer series for the pressure exist in the thermodynamic limit below certain thresholds in the temperature. Our methods apply also to correlation functions and yield bounds on the asymptotic behavior of their Mayer coefficients.

KEY WORDS: Coulomb systems; Mayer series; two-dimensional systems; non-charge-symmetric systems.

1. INTRODUCTION

In this paper we study the thermodynamic limit of the coefficients of the Mayer series for the pressure—or, more generally, for correlation functions—for neutral systems of particles in two dimensions, interacting by Coulomb forces, and with a hard core to prevent ultraviolet catastrophe. This problem (for the pressure) has been discussed thoroughly in Ref. 1 for the case of a system of two species of opposite charges, and that discussion would extend immediately to the case of a charge-symmetric system. The authors find a simple behavior governed by the value of the parameter $\Gamma = \beta e^2$, with β the inverse temperature and e the charge: (i) all coefficients appear to diverge for $\Gamma \leqslant 2$; (ii) there is a series of thresholds between $\Gamma = 2$ and $\Gamma = 4$, such that above the Nth threshold $\Gamma_N \equiv 4 - 2/N$, the coefficient of order $2N$ is finite; (iii) all coefficients are finite above $\Gamma = 4$, the Kosterlitz–Thouless transition point. These results have recently been extended[(2)] to show that the series is in fact asymptotic to the pressure, above slightly higher thresholds, to the order in which the coefficients are finite.

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In the current work we remove the restriction of charge symmetry, allowing an arbitrary collection of species of charged particles which can form neutral systems. (We adopt some notation from Lieb and Lebowitz[3] for such systems.) Our general approach is also somewhat different from that of Ref. 1, which was characterized by the decomposition of the Coulomb interaction as an infinite sum of short-range interactions on various scales, the use of the sine-Gordon transformation, and the treatment of the neutral system as a limit of nonneutral systems. Here we study directly in configuration space the integrals which define the Mayer coefficients. In either case, the problem is to exhibit the two levels of cancellations which are necessary for finiteness. At the first level, cancellations take place between various terms in the logarithm of the partition function, a familiar mechanism which in Ref. 1 leads to the usual reorganization of the series into a sum over connected graphs, but for us must be handled somewhat differently since we work directly with the long-range interaction. A second level of cancellations is also necessary in these Coulomb systems. In Ref. 1 they are produced when the contribution from a fixed configuration of particles is summed over all possible charge assignments to these particles. This does not work in the non-charge-symmetric case considered here (the difference is not simply one of methodology). We instead improve the falloff in the interaction between neutral clusters by averaging over their orientations, although this leads to technical difficulties as rotating clusters hit the boundary of the box within which the system is confined.

We find thresholds in the temperature, similar to those of Ref. 1, below which the coefficients in the Mayer series are finite. These may be characterized as follows: the coefficient of the pressure which corresponds to a specific collection of particles is finite if, for any decomposition of the collection into nonneutral subcollections, the modified canonical partition function defined by integrating only over configurations in which these clusters have bounded size, and in which all intercluster distances are the same order of magnitude, is finite. (In the charge symmetric case[1] this finiteness is guaranteed by finiteness when all subcollections consist of a single particle, leading to the threshholds described above.) Our methods also apply to correlation functions and lead to bounds on the asymptotic behavior in configuration space of the coefficients of their Mayer series, as we discuss briefly in Section 7.

2. THE MODEL

We consider two-dimensional systems assembled from S species of particles, having nonzero charges $e^1,..., e^S$, activities $z^1,..., z^S$, and hard core diameter d, and interacting via the Coulomb potential. (Inclusion of neutral particles with hard cores or of distinct diameters for different species would cause minor technical difficulties; see the remark following Lemma 3.6.) A system with N^s particles of species s has charge $\mathbf{N} \cdot \mathbf{e} = \sum_s N^s e^s$. We treat only neutral systems ($\mathbf{N} \cdot \mathbf{e} = 0$) and hence assume that the charges are such that neutral systems can form which contain any given species. We write $N = \sum_s N^s$ and for each $\mathbf{N}$ choose once and for all an indexing of the particles in the system by $I_N = \{1,..., N\}$, writing e_i, z_i, and x_i for the charge, activity, and position of the ith particle.

For fixed $\mathbf{N}$ and $A \subset I_N$ let $e(A)\ (= e_{\mathbf{N}}(A)) = \sum_{i \in A} e_i$. The energy of the particles indexed by A is

$$U_{\mathbf{N}}(A, x) = -\sum_{\substack{i,j \in A \\ i<j}} e_i e_j \log\left(\frac{|x_i - x_j|}{a}\right)$$

By an electrostatic argument,[4] for $e(A) = 0$,

$$U_{\mathbf{N}} \geqslant -b\,|A| \tag{2.1}$$

for some constant b, as long as $|x_i - x_j| \geqslant d$. Set

$$\begin{aligned} u_{\mathbf{N}}(A; x) &= \exp[-\beta U_{\mathbf{N}}(A; x)] \\ &= a^{(\beta/2)\sum_{i \in A} e_i^2} \prod_{\substack{i,j \in A \\ i<j}} |x_i - x_j|^{\beta e_i e_j} \end{aligned}$$

Then the grand canonical partition function for neutral systems in the volume Λ is

$$\Xi(\beta, \mathbf{z}; \Lambda) = \sum_{\substack{\mathbf{N} \geqslant 0 \\ \mathbf{e} \cdot \mathbf{N} = 0}} \frac{\mathbf{z}^{\mathbf{N}}}{\mathbf{N}!} \int_{\Lambda^N} u_{\mathbf{N}}(I_N; x)\, \chi_{\{|x_i - x_j| \geqslant d\}}(x)\, dx_1 \cdots dx_N \tag{2.2}$$

where the series converges for small activities by the stability condition (2.1). Since a change in length scale $a \to \lambda a$ corresponds to a change in activities $z_i \to \lambda^{-\beta e_i^2/2} z_i$, we may without loss of generality set $a = 1$ in the sequel.

We call a set $A \subset I_N$ *neutral* if $e(A) = 0$, and a partition $\mathscr{P}$ of I_N *neutral* if $e(P) = 0$ for each set $P \in \mathscr{P}$. Then we have the following:

Lemma 2.1. The grand canonical pressure in Λ has series expansion

$$\begin{aligned} \pi(\beta, \mathbf{z}; \Lambda) &\equiv |\Lambda|^{-1} \log \Xi(\beta, \mathbf{z}; \Lambda) \\ &= \sum_{\substack{\mathbf{N} \geqslant 0 \\ \mathbf{e} \cdot \mathbf{N} = 0}} \frac{\mathbf{z}^{\mathbf{N}}}{\mathbf{N}!} |\Lambda|^{-1} \int_{\Lambda^N} F_{\mathbf{N}}(\beta, x)\, dx_1 \cdots dx_n \end{aligned} \tag{2.3}$$

with

$$F_{\mathbf{N}}(\beta, x) = \sum_{\mathscr{P}} (-1)^{|\mathscr{P}|-1}(|\mathscr{P}|-1)!\, f_{\mathscr{P}}(\beta, x)\, \psi_{\mathscr{P}}(x) \tag{2.4}$$

and

$$f_{\mathscr{P}}(\beta, x) = \prod_{P \in \mathscr{P}} u_{\mathbf{N}}(P, x) \tag{2.5}$$

$$\psi_{\mathscr{P}}(x) = \prod_{P \in \mathscr{P}} \prod_{\substack{i,j \in P \\ i \neq j}} \chi_{\{|x_i - x_j| \geq d\}}(x) \tag{2.6}$$

Here the sum is over all neutral partitions $\mathscr{P}$.

Proof. From (2.2),

$$\log \Xi = \sum_{k=1}^{\infty} (-1)^{k-1}(k-1)! \sum_{\substack{\mathbf{N}_1,\ldots,\mathbf{N}_k \\ \mathbf{N}_i \geq 0; \mathbf{e}\cdot\mathbf{N}=0}} \frac{\mathbf{z}^{\Sigma \mathbf{N}_i}}{k!\, \mathbf{N}_1! \cdots \mathbf{N}_k!}$$

$$\times \int \prod_{i=1}^{k} u_{\mathbf{N}_i}(I_{\mathbf{N}_i}, x)\, dx \tag{2.7}$$

Set $\mathbf{N} = \sum_{i=1}^{k} \mathbf{N}_i$. Then $\binom{\mathbf{N}}{\mathbf{N}_1 \cdots \mathbf{N}_k}$ counts the number of ordered partitions of I_N into k sets $P_1,\ldots, P_k$, with P_i containing N_i^s particles of species s, and thus (2.7) may be rewritten as a sum over ordered partitions. The factor $(k!)^{-1}$ converts this into a sum over (unordered) partitions. ∎

We can now state our main result on the existence of the thermodynamic limit for the coefficients of the pressure.

Theorem 2.2. Fix $\mathbf{N}$. Suppose that β is sufficiently large that

$$\varepsilon \equiv 2 - \sup_{\mathscr{D}} \sum_{A \in \mathscr{D}, e(A) \neq 0} \left[2 - \frac{\beta}{2} e(A)^2 \right] > 0 \tag{2.8}$$

where the supremum is over all partitions $\mathscr{D}$ of I. Then if Λ_ρ is a ball of radius ρ,

$$\pi_{\mathbf{N}}(\beta) \equiv \lim_{\rho \to \infty} |\Lambda_\rho|^{-1} \int_{\Lambda_\rho^N} F_{\mathbf{N}}(\beta, x)\, dx \tag{2.9}$$

exists.

Remark 2.3. If there exist only two species of particle, with charges $\pm e$, then (2.8) is equivalent to

$$\beta e^2 > 4\left(1 - \frac{1}{N}\right) \tag{2.10}$$

which is the convergence condition of Ref. 1. Moreover, if all charges e^s are integer multiples of some fundamental charge e, (2.10) is sufficient for (2.8); in particular, all coefficients $\pi_N(\beta)$ exist for $\beta e^2 \geqslant 4$.

We close this section with a discussion of the origin of the condition (2.8) and a broad outline of the proof of the theorem. Let us replace the limit (2.9) with the formally equivalent integral

$$\int_{(\mathbb{R}^2)^{N-1}} F_N(\beta, x)\, dx_2 \cdots dx_n \tag{2.11}$$

and consider only the $\mathscr{P} = \{I\}$ contribution to F_N in (2.11). Let $\mathscr{D}$ be a partition of I; we integrate (2.11) over configurations in which particles indexed by each subset $A \in \mathscr{D}$ form a cluster of bounded size, and in which all intercluster distances are order of magnitude R. The integral looks like

$$\int^{\infty} R^{(\beta/2)\sum_{A,B \in \mathscr{D}} e(A) e(B)} R^{2(|\mathscr{D}|-1)} \frac{dR}{R}$$

which converges if

$$\sum_{A \in \mathscr{D}} \left[2 - \frac{\beta}{2} e(A)^2 \right] < 2 \tag{2.12}$$

Now (2.12) can certainly not hold at any β for all $\mathscr{D}$—in particular, not for neutral partitions. To show that in fact the weaker condition (2.8) suffices for the thermodynamic limit [i.e., at least formally, for the convergence of (2.11)] we will proceed as follows:

(i) Inclusion of all terms in $\sum_{\mathscr{P}}$ in (2.1) yields cancellations (as for the usual Mayer expansion), which provide falloff in $F_N(\beta, x)$ as neutral clusters separate.

(ii) Even so, (2.11) is not absolutely convergent, but rather exists for an appropriate order of integration in which we first average over orientations of neutral clusters, leaving an absolutely convergent integration over the remaining coordinates.

(iii) To display the cancellations of (i) and (ii) we subdivide configuration space into regions within which the particles are grouped hierarchically into well-defined clusters.

(iv) Finally, we must treat the actual thermodynamic limit (2.9) rather than the formally equivalent integral (2.11); the boundary of Λ causes difficulty by interfering with the averaging over cluster orientation. The solution is the further subdivision of configuration space into subregions; within a given subregion each cluster either may be averaged over orientations or is confined to lie near the boundary. The contribution of any subregion involving a cluster of the latter type vanishes in the thermodynamic limit. It is at this point that we need the box Λ to be a disk.

The details of the argument are presented in Sections 3–6. In Section 3 we give the decomposition of configuration space into regions and subregions as discussed above. In Section 4 we state two basic lemmas estimating, respectively, the integrand within any subregion and the integral over that subregion, and from these give the proof of Theorem 2.2. The lemmas are proved in Sections 5 and 6.

3. CLUSTERS

In this section we fix $N>0$ and subdivide the configuration space of N particles into regions in which the particles are grouped hierarchically into clusters. Our definition is somewhat arbitrary since it depends on the order in the index set $I=\{1,..., N\}$; in particular, if $A\subset I$, we let i_A denote the smallest element of A and treat the particle indexed by i_A as the center of the cluster indexed by A.

Definition 3.1. (a) A hierarchy $\mathscr{H}$ is a collection of subsets of I called *clusters* such that (i) $I\in\mathscr{H}$, (ii) if $A\in\mathscr{H}$, then $|A|\geqslant 2$, and (iii) if $A, B\in\mathscr{H}$, then $A\subset B$, $B\subset A$, or $A\cap B=\varnothing$.

(b) If $\mathscr{H}$ is a hierarchy we define $\bar{\mathscr{H}}=\mathscr{H}\cup\{\{1\},..., \{N\}\}$ (corresponding to the treatment of a single particle as a trivial cluster).

(c) For $A\in\mathscr{H}$ we define $\mathscr{H}_0(A)$ to be the family of maximal proper subsets of A in $\bar{\mathscr{H}}$, $A_*\in\mathscr{H}_0(A)$ to satisfy $i_{A_*}=i_A$, and $\mathscr{H}_1(A)=\mathscr{H}_0(A)\backslash\{A_*\}$. Finally, for $A\in\bar{\mathscr{H}}$ with $A\neq I$ we let A^* be the minimal proper superset of A in $\mathscr{H}$.

To each point $x\in(\mathbb{R}^2)^N$ we will associate (uniquely for a.e. x) a hierarchy $\mathscr{H}$ characterized by the fact that interparticle distances within a cluster $A\in\mathscr{H}$ are an order of magnitude less than the distance from A to any disjoint cluster.

Definition 3.2. (a) Suppose that $x\in\mathbb{R}^{2N}$ and that $\mathscr{H}$ is a hierarchy. For $A\in\mathscr{H}$ we measure the size of A by

$$r_A(\mathscr{H}; x)=\sup_{B\in\mathscr{H}_0(A)}|x_{i_A}-x_{i_B}|$$

and the separation of A from other clusters by

$$R_A(\mathscr{H}; x)=\inf_{\substack{B\in\mathscr{H}_0(A^*)\\ B\neq A}}|x_{i_A}-x_{i_B}|$$

with $R_I(\mathscr{H}; x) = \infty$ by convention. We frequently write $r_A(\mathscr{H}; x) \equiv r_A$, etc., when no confusion can arise.

(b) Let α satisfy $0 < \alpha < 1/7$. Then if $\mathscr{H}$ is a hierarchy we define regions $\tilde{X}_{\mathscr{H}}$ and $X_{\mathscr{H}}$ in $\mathbb{R}^{2N}$ by

$$\tilde{X}_{\mathscr{H}} = \{x \mid r_A(\mathscr{H}; x) < \alpha R_A(\mathscr{H}; x), \text{ all } A \in \mathscr{H}\}$$

$$X_{\mathscr{H}} = \tilde{X}_{\mathscr{H}} \Big\backslash \Big(\bigcup_{\mathscr{H}' \supset \mathscr{H}} \tilde{X}_{\mathscr{H}'} \Big)$$

(Thus for $x \in \tilde{X}_{\mathscr{H}}$ the particles form clusters described by $\mathscr{H}$; for $x \in X_{\mathscr{H}}$, only these clusters are formed.)

We now give some elementary properties of these regions.

Lemma 3.3. For $x \in \tilde{X}_{\mathscr{H}}$ and $A \in \mathscr{H}$,

(a) $r_A \leqslant \alpha r_{A*}$;

(b) If $i \in A$ then $|x_{I_A} - x_i| < (1-\alpha)^{-1} r_A$;

(c) If $i, j \in A$ then $|x_i - x_j| \leqslant 2(1-\alpha)^{-1} r_A$;

(d) If $i \in A$ and $j \notin A$ then $|x_i - x_j| \geqslant (1-3\alpha)(1-\alpha)^{-1} R_A$.

Proof. (a) is immediate, (b) follows from (a) by induction on the subsets of A and in turn implies (c). To verify (d) let B be the minimal element of $\mathscr{H}$ containing i and j and let $C, D \in \mathscr{H}_0(B)$ satisfy $i \in C$, $j \in D$. Then since $R_C, R_D \leqslant |x_{i_C} - x_{i_D}|$,

$$\begin{aligned} |x_i - x_j| &\geqslant |x_{i_C} - x_{i_D}| - |x_{i_C} - x_i| - |x_{i_D} - x_j| \\ &\geqslant [1 - 2\alpha(1-\alpha)^{-1}]\, |x_{i_C} - x_{i_D}| \\ &\geqslant (1-3\alpha)(1-\alpha)^{-1} R_A. \quad \blacksquare \end{aligned}$$

Lemma 3.4. (a) Suppose $x \in \tilde{X}_{\mathscr{H}} \cap \tilde{X}_{\mathscr{H}'}$ has all coordinates $x_i \in \mathbb{R}^2$ distinct. Then $\mathscr{H} \cup \mathscr{H}'$ is a hierarchy and $x \in \tilde{X}_{\mathscr{H} \cup \mathscr{H}'}$.

(b) Each $x \in \mathbb{R}^{2N}$ with distinct coordinates belongs to a unique region $X_{\mathscr{H}}$.

Proof. (a) The sets in $\mathscr{H} \cup \mathscr{H}'$ are nonoverlapping since if $A \in \mathscr{H}$, $B \in \mathscr{H}'$, and $i \in A \backslash B$, $j \in B \backslash A$, $k \in A \cap B$, then from (c), (d) of Lemma 3.3,

$$|x_i - x_j| \leqslant 2\alpha(1-\alpha)^{-1}[R_A(\mathscr{H}; x) + R_B(\mathscr{H}'; x)]$$

and

$$|x_i - x_j| \geqslant (1-3\alpha)[2(1-\alpha)]^{-1}[R_A(\mathscr{H}; x) + R_B(\mathscr{H}'; x)]$$

a contradiction for $\alpha < 1/7$. Hence for $A \in \mathscr{H}$,

$$r_A(\mathscr{H} \cup \mathscr{H}'; x) \leqslant r_A(\mathscr{H}; x) \leqslant \alpha R_A(\mathscr{H}; x) \leqslant \alpha R_A(\mathscr{H} \cup \mathscr{H}'; x)$$

and similarly for $A \in \mathscr{H}'$, so that $x \in \tilde{X}_{\mathscr{H} \cup \mathscr{H}'}$.

(b) From (a) we see that (since $\tilde{X}_{\{I\}} = \mathbb{R}^{2N}$) each x with distinct components belongs to $\tilde{X}_{\mathscr{H}}$ for a unique maximal $\mathscr{H}$, so that $x \in X_{\mathscr{H}}$ uniquely. ▌

Lemma 3.5. If $x \in X_{\mathscr{H}}$, $A \in \mathscr{H}$, and $B, C \in \mathscr{H}_0(A)$ with $B \neq C$, then

$$|x_{i_B} - x_{i_C}| \geqslant \left(\frac{\alpha}{1+\alpha}\right)^{N-2} r_A(\mathscr{H}; x)$$

Proof. Let $|\mathscr{H}_0(A)| = k$, and for $2 \leqslant j \leqslant k$ consider sets $D \subset A$ which are unions of j sets from $\mathscr{H}_0(A)$, so that $\mathscr{H} \cup \{D\}$ is a hierarchy. Let $u_j = \inf_D r_D(\mathscr{H} \cup \{D\}; x)$ and suppose that this infimum is realized by $D = D_j$. If $j < k$ and $C \in \mathscr{H}_0(A)$ with $C \cap D_j = \varnothing$, then

$$\begin{aligned} u_{j+1} &\leqslant \inf_C r_{D_j \cup C}(\mathscr{H} \cup \{D_j \cup C\}; x) \\ &\leqslant \inf_C [|x_{i_D} - x_{i_C}| + r_{D_j}(\mathscr{H} \cup \{D_j\}; x)] \\ &= R_{D_j}(\mathscr{H} \cup \{D_j\}; x) + r_{D_j}(\mathscr{H} \cup \{D_j\}; x) \end{aligned} \tag{3.1}$$

On the other hand, $x \notin \tilde{X}_{\mathscr{H} \cup \{D_j\}}$ since $2 \leqslant j < k$, from which

$$R_{D_j}(\mathscr{H} \cup \{D_j\}; x) < \alpha^{-1} r_{D_j}(\mathscr{H} \cup \{D_j\}; x)$$

Thus (3.1) becomes $u_{j+1} \leqslant [(\alpha+1)/\alpha]\, u_j$, so that $u_2 \geqslant [\alpha/(\alpha+1)]^{k-2}\, u_k$, which implies the lemma. ▌

We next observe that the regions $X_{\mathscr{H}}$ are invariant under rotations of entire clusters. Let $S(\theta)$: $\mathbb{R}^2 \to \mathbb{R}^2$ denote rotation by angle θ, and for $A \subset I$ define $S_A(\theta)$: $\mathbb{R}^{2N} \to \mathbb{R}^{2N}$ by

$$(S_A(\theta)\, x)_i = \begin{cases} x_i, & \text{if } i \notin A \\ x_{i_A} + S(\theta)(x_i - x_{i_A}), & \text{if } i \in A \end{cases}$$

Lemma 3.6. If $x \in X_{\mathscr{H}}$ and $A \in \mathscr{H}$, then $S_A(\theta)\, x \in X_{\mathscr{H}}$ for any θ. Moreover, for any partition $\mathscr{P}$ with $A \cap P$ neutral for all $P \in \mathscr{P}$, $\psi_{\mathscr{P}}(x) = \psi_{\mathscr{P}}(S_A(\theta)\, x)$ [see (2.6)].

Proof. Let $\tilde{x} = S_A(\theta)\, x$. If the hierarchy $\mathscr{H}'$ contains A we have $r_C(\mathscr{H}'; x) = r_C(\mathscr{H}'; \tilde{x})$ and $R_C(\mathscr{H}'; x) = R_C(\mathscr{H}'; \tilde{x})$ for any $C \in \mathscr{H}'$; from this the first statement follows. Suppose now that, say, $\psi_{\mathscr{P}}(\tilde{x})=0$, i.e., $|\tilde{x}_i-\tilde{x}_j|<d$ for some $i, j\in P\in\mathscr{P}$. Then certainly $\psi_{\mathscr{P}}(x)=0$ also unless $i\in A$, $j\notin A$. But since $A\cap P$ is neutral there is a $k\in A\cap P$ with $k\neq i$; then by Lemma 3.3,

$$\begin{aligned} d &\geqslant (1-3\alpha)(1-\alpha)^{-1}\alpha^{-1}r_A(\mathscr{H};x)\geqslant(2\alpha)^{-1}(1-3\alpha)\,|x_i-x_k| \\ &\geqslant |x_i-x_k| \end{aligned}$$

so that $\psi_{\mathscr{P}}(x)=0$ in this case also. ∎

We remark that the second statement of the lemma would fail if distinct species had distinct hard core diameters or if neutral species occurred. This would necessitate special treatment of small clusters in what follows, but would not affect the results.

Finally we introduce the further decomposition of Λ^N needed to control the effect of the boundary by associating to each $x\in X_{\mathscr{H}}\cap\Lambda$ a subfamily $\mathscr{A}\subset\mathscr{H}$ of clusters which (essentially) cannot be rotated freely without some point leaving Λ.

Definition 3.7. (a) For any $A\subset I$ and $x\in\Lambda^N$ define $s_A(\Lambda;x)$ to be the distance from x_{i_A} to the boundary of Λ. Then for any hierarchy $\mathscr{H}$ and subfamily $\mathscr{A}\subset\mathscr{H}$ let $X_{\mathscr{H},\mathscr{A}}(\Lambda)\subset\mathbb{R}^{2N}$ be the set of all $x\in X_{\mathscr{H}}\cap\Lambda$ such that, for all $A\in\mathscr{H}$,

$$A\in\mathscr{A}\Leftrightarrow r_A(\mathscr{H};x)\geqslant(1-\alpha)\,s_A(\Lambda;x) \tag{3.2}$$

(b) A subfamily $\mathscr{A}$ of a hierarchy $\mathscr{H}$ is *ancestral* if every superset in $\mathscr{H}$ of a set in in $\mathscr{A}$ is also in $\mathscr{A}$.

Lemma 3.8. (a) $X_{\mathscr{H},\mathscr{A}}(\Lambda)$ is empty unless $\mathscr{A}$ is ancestral.

(b) The regions $X_{\mathscr{H},\mathscr{A}}(\Lambda)$ with $\mathscr{H}$ any hierarchy and $\mathscr{A}\subset\mathscr{H}$ ancestral partition Λ^N, up to a set of measure zero.

(c) If $x\in X_{\mathscr{H},\mathscr{A}}(\Lambda)$ and $B\in\mathscr{H}\backslash\mathscr{A}$, then $S_B(\theta)\,x\in X_{\mathscr{H},\mathscr{A}}(\Lambda)$ for any θ.

Proof. (a) If $x\in X_{\mathscr{H},\mathscr{A}}(\Lambda)$ and $A\subset\mathscr{A}$, then

$$\begin{aligned} s_{A^*}(\Lambda;x) &\leqslant |x_{i_{A^*}}-x_{i_A}|+s_A(\Lambda;x) \\ &\leqslant r_{A^*}(\mathscr{H};x)[1+\alpha(1-\alpha)^{-1}] \\ &=(1-\alpha)^{-1}\,r_{A^*}(\mathscr{H};x) \end{aligned}$$

so $A^*\in\mathscr{A}$. Thus $\mathscr{A}$ is ancestral.

(b) This follows immediately from (a) and Lemma 3.4(b).

(c) Let $\tilde{x}=S_B(\theta)\,x$. Now $\tilde{x}_i=x_i\in\Lambda$ unless $i\in B$. But for $i\in B$, Lemma 3.3 and the hypothesis $B\notin\mathscr{A}$ imply

$$|\tilde{x}_i-x_{i_B}|\leqslant(1-\alpha)^{-1}\,r_B(\mathscr{H};x)<s_B(\Lambda;x)$$

so again $\tilde{x}_i\in\Lambda$. We know $\tilde{x}\in X_{\mathscr{H}}$ by Lemma 3.6, so it remains to verify (3.2) for $\tilde{x}$. Since $r_A(\mathscr{H};x)=r_A(\mathscr{H};\tilde{x})$ for all $A\in\mathscr{H}$, and $s_A(\Lambda;x)=s_A(\Lambda;\tilde{x})$ unless $A\subsetneqq B$, (3.2) could fail only for $A\subsetneqq B$. But this is in turn impossible (since $B\notin\mathscr{A}$) by (a). ∎

4. PROOF OF THE MAIN THEOREM

In this section we give two estimates—Lemmas 4.3 and 4.4, to be proved in Sections 5 and 6—and from these prove Theorem 2.2.

We first look again at the simpler problem of the convergence of (2.11): for a fixed neutral partition $\mathscr{P}$ and hierarchy $\mathscr{H}$ consider

$$\int_{X_{\mathscr{H}}(x_1)} f_{\mathscr{P}}\psi_{\mathscr{P}}\,dx_2\cdots dx_N \tag{4.1}$$

where $X_{\mathscr{H}}(x_1)$ is the section $\{(x_2,\ldots,x_N)\,|\,(x_1,x_2,\ldots,x_N\}\in X_{\mathscr{H}}\}$. For $i,j\in I$ let A_{ij} be the minimal element of $\mathscr{H}$ containing i and j. Now if $x\in X_{\mathscr{H}}$, Lemmas 3.3 and 3.5 imply

$$c_1 r_{A_{ij}}(\mathscr{H};x)\leqslant|x_i-x_j|\leqslant c_2 r_{A_{ij}}(\mathscr{H};x)$$

for some $0<c_1\leqslant c_2<\infty$. Thus

$$\begin{aligned} f_{\mathscr{P}}\psi_{\mathscr{P}}&\leqslant C\prod_{P\in\mathscr{P}}\left(\prod_{A\in\mathscr{H}} r_A^{(\beta/2)\Sigma_{ij}e_ie_j}\right)\psi_{\mathscr{P}}(x)\\ &\leqslant C\prod_{A\in\mathscr{H}} r_A^{\kappa_A-\Sigma_A\kappa_B-2|\mathscr{H}_1(A)|}\chi_{\{r_A\geqslant\delta_A\}} \end{aligned}\tag{4.2}$$

where in the first line $\sum_{ij}\equiv\sum_{\{i,j\in\mathscr{P}\,|\,A_{ij}=A\}}$ and in the second we have introduced the general notation

$$\textstyle\sum_A\kappa_B\equiv\sum_{B\in\mathscr{H}_0(A)}\kappa_B \tag{4.3}$$

and defined

$$\begin{aligned} \kappa_A\equiv\kappa_A(\mathscr{P})&=\sum_{i\in A}\left(2-\frac{\beta}{2}e_i^2\right)+\frac{\beta}{2}\sum_{P\in\mathscr{P}}e(P\cap A)^2-2\\ \delta_A\equiv\delta_A(\mathscr{P})&=\begin{cases}0, & \text{if } |P\cap A|\leqslant 1,\ \text{all } P\in\mathscr{P}\\ d, & \text{otherwise}\end{cases} \end{aligned}\tag{4.4}$$

Change variables in (4.1) by defining, for $B \in \mathscr{H}_1(A)$

$$x_{i_B} = x_{i_A} + r_A \xi_{AB} \tag{4.5a}$$

with

$$\max_{B \in \mathscr{H}_1(A)} |\xi_{AB}| = 1 \tag{4.5b}$$

insert (4.2), and integrate over all ξ_{AB}. Thus we find that (4.1) is bounded by

$$C \prod_{A \in \mathscr{H}} \int_{\delta_A}^{\alpha r_{A^*}} r_A^{\kappa_A - \Sigma_A \kappa_B} \frac{dr_A}{r_A} \tag{4.6}$$

where $r_{I^*} = \infty$ by convention.

Now (4.6) is easily estimated recursively (see proof of Lemma 6.2, Case 1); it converges if for any partition $\mathscr{D}$ of I with $\mathscr{D} \subset \bar{\mathscr{H}}$ (or equivalently, since $\kappa_A = 0$ if $|A| = 1$, for any subfamily $\mathscr{D} \subset \mathscr{H}$ of pairwise disjoint sets) with $\mathscr{D} \neq \{I\}$,

$$\kappa_I - \sum_{A \in \mathscr{D}} \kappa_A < 0 \tag{4.7}$$

But the left side of (4.7) is

$$\sum_{A \in \mathscr{D}} \left[2 - \frac{\beta}{2} \sum_{P \in \mathscr{P}} e(P \cap A)^2 \right] - 2 \tag{4.8}$$

When the condition (2.8) is satisfied, (4.8) is negative unless there exist sets $A \subset \mathscr{H}$ with $P \cap A$ neutral for all $P \in \mathscr{P}$. Moreover, we see that (4.1) would converge if, in the estimate (4.2), κ_A could be replaced by $\kappa_A + 2$ for such A. Lemma 4.3 below shows that an improvement from κ_A to $\kappa_A + 1$ can be achieved by summing over $\mathscr{P}$, and a further improvement to $\kappa_A + 2$ by averaging over orientation of neutral clusters.

Definition 4.1. A neutral hierarchy $\mathscr{N}$ is a hierarchy in which each cluster is neutral. For $\mathscr{H}$ any hierarchy we let $\mathscr{N}(\mathscr{H})$ denote the neutral hierarchy consisting of all neutral clusters in $\mathscr{H}$; for $\mathscr{P}$ a neutral partition we let $v(\mathscr{P}) (= v_{\mathscr{H}}(\mathscr{P})) \subset \mathscr{N}(\mathscr{H})$ denote the neutral hierarchy of clusters $A \subset \mathscr{H}$ which satisfy $e(A \cap P) = 0$ for all $P \in \mathscr{P}$.

Definition 4.2. For $A \subset I$ we define the averaging operator K_A, acting on functions on $\mathbb{R}^{2N}$, by

$$(K_A f)(x) = (2\pi)^{-1} \int_0^{2\pi} f(S_A(\theta)\, x)\, d\theta$$

Lemma 4.3. Let $\mathscr{H}$ be a hierarchy, $\mathscr{A} \subset \mathscr{N}(\mathscr{H})$ an arbitrary subfamily. Then for $x \in X_{\mathscr{H},\mathscr{A}}$ we may write

$$\left(\prod_{A \in \mathscr{N}(\mathscr{H})\setminus\mathscr{A}} K_A\right) F_N(\beta, x) = \sum_{\mathscr{P}} g_{\mathscr{P},\mathscr{H},\mathscr{A}}(\beta, x) \tag{4.9}$$

with

$$\begin{aligned} |g_{\mathscr{P},\mathscr{H},\mathscr{A}}(\beta, x)| &\leqslant G_{\mathscr{P},\mathscr{H},\mathscr{A}}(\beta, x) \\ &\equiv C \prod_{A \in \mathscr{H}} r_A(\mathscr{H}; x)^{\mu_A - \sum_A \mu_B - 2|\mathscr{H}_1(A)|} \chi_{\{r_A(\mathscr{H};x) \geqslant \delta_A\}} \end{aligned} \tag{4.10}$$

Here $\mu_A(=\mu_A(\mathscr{P}, \mathscr{A})) = \kappa_A(\mathscr{P}) + n_A(\mathscr{P}, \mathscr{A})$, where

$$n_A(\mathscr{P}, \mathscr{A}) = \begin{cases} 2, & \text{if } A \in v(\mathscr{P})\setminus\mathscr{A},\ A \neq I \\ 1, & \text{if } A \in v(\mathscr{P}) \cap \mathscr{A},\ A \neq I \\ 0, & \text{otherwise} \end{cases}$$

Having estimated the integrands, we may estimate the integrals.

Lemma 4.4. Let $\mathscr{H}$ be a hierarchy, $\mathscr{A} \subset \mathscr{H}$ an ancestral subfamily, Λ a disk. Then for any ε' with $0 < \varepsilon' < \varepsilon$:

(a) If $\mathscr{A} = \varnothing$,

$$\int_{\substack{X_{\mathscr{H}}(x_1) \\ r_I(\mathscr{H};x) \geqslant u}} G_{\mathscr{P},\mathscr{H},\varnothing}\, dx_2 \cdots dx_N \leqslant C u^{-\varepsilon'}$$

the integral is of course independent of u for $u < d$.

(b) If $\mathscr{A} \neq \varnothing$,

$$\int_{X_{\mathscr{H},\mathscr{A}}} G_{\mathscr{P},\mathscr{H},\mathscr{A}}\, dx_1 \cdots dx_N \leqslant C\, |\Lambda|^{1-\varepsilon'/2}$$

Proof of Theorem 2.2. From Lemmas 4.3 and 4.4, and the Lebesque dominated convergence theorem, $g_{\mathscr{P},\mathscr{H},\varnothing}$ is, for fixed x_1, absolutely integrable in $x_2,..., x_N$ over $X_{\mathscr{H}}(x_1)$. Set

$$\pi_{\mathbf{N}}(\beta) = \sum_{\mathscr{H},\mathscr{P}} \int_{X_{\mathscr{H}}(x_1)} g_{\mathscr{P},\mathscr{H},\varnothing}\, dx_2 \cdots dx_N$$

$\pi_{\mathbf{N}}$ is of course independent of x_1. Then we write

$$\int_{\Lambda^N} F_{\mathbf{N}}\, dx = \sum_{\mathscr{H},\mathscr{A}} \int_{X_{\mathscr{H},\mathscr{A}}} F_{\mathbf{N}}\, dx = \sum_{\mathscr{H},\mathscr{A},\mathscr{P}} \int_{X_{\mathscr{H},\mathscr{A}}} g_{\mathscr{P},\mathscr{H},\mathscr{A}}\, dx$$

where the sum is over hierarchies, their ancestral subfamilies, and neutral partitions, and we have used Lemmas 3.8 and 4.3. Noting that

$$X_{\mathscr{H},\varnothing} = \{x \in X_{\mathscr{H}} \mid x_1 \in A,\, r_I(\mathscr{H}; x) \leqslant (1-\alpha)\, s_I(A; x)\}$$

we have

$$\begin{aligned}
&\left| |A_\rho|^{-1} \int_{A_\rho^N} F_{\mathbf{N}}\, dx - \pi_{\mathbf{N}} \right| \\
&\quad \leqslant |A_\rho|^{-1} \sum_{\mathscr{H},\mathscr{P}} \int_{A_\rho} dx_1 \Bigg(\int_{\substack{X_{\mathscr{H}} \\ r_I \geqslant (1-\alpha)s_I}} G_{\mathscr{P},\mathscr{H},\varnothing}\, dx_2 \cdots dx_N \\
&\qquad + \sum_{\mathscr{A} \neq \varnothing} \int_{X_{\mathscr{H},\mathscr{A}}(x_1)} G_{\mathscr{P},\mathscr{H},\mathscr{A}}\, dx_2 \cdots dx_N \Bigg) \\
&\quad \leqslant |A_\rho|^{-1} \left\{ C_1 \int_{\substack{A_\rho \\ (1-\alpha)s_I \geqslant d}} [(1-\alpha)\, s_I]^{-\varepsilon'}\, dx_1 + C_2\, |A_\rho|^{1-\varepsilon'/2} \right\} \\
&\quad = C_1' \int_0^{1-d/\rho(1-\alpha)} u[\rho(1-\mu)]^{-\varepsilon'}\, du + C_2\, |A_\rho|^{-\varepsilon'/2}
\end{aligned}$$

where $u = |x|/\rho$; the limit as $\rho \to \infty$ is zero by the Lebesque dominated convergence theorem. ∎

5. ESTIMATING THE INTEGRAND

In this section we prove Lemma 4.3 by expanding $F_{\mathbf{N}}(\beta, x)$ in a generalized Taylor series[5] in variables which scale the size of each neutral cluster in a hierarchy. The sum over neutral partitions $\mathscr{P}$ in (2.4) then cancels the terms of order zero and an averaging over orientations the terms of order one; (4.10) then arises by bounding the remainder.

The next definition should be read as an extension of Definition 4.1.

Definition 5.1. (a) for $\mathscr{P}$ and $\mathscr{Q}$ neutral partitions we write $\mathscr{P} \prec \mathscr{Q}$ if $\mathscr{P}$ refines $\mathscr{Q}$, and write $\mathscr{P} \wedge \mathscr{Q}$ for their coarsest common refinement when it is neutral; otherwise, $\mathscr{P} \wedge \mathscr{Q}$ is undefined.

(b) If $\mathscr{N}$ is a neutral hierarchy we let $\pi(\mathscr{N})$ denote the neutral partition formed by all nonempty sets of the form $A/\mathscr{N} \equiv A \backslash (\bigcup_{B \in \mathscr{N}_0(A)} B)$, for $A \in \mathscr{N}$.

We observe that

$$\pi(\mathscr{N}_1) \prec \pi(\mathscr{N}_2) \qquad \text{whenever} \quad \mathscr{N}_1 \supset \mathscr{N}_2 \tag{5.1}$$

and that if $\mathscr{H}$ is a fixed hierarchy and $\nu = \nu_{\mathscr{H}}$,

$$\nu(\mathscr{P}_1) \subset \nu(\mathscr{P}_2) \qquad \text{whenever} \quad \mathscr{P}_1 \prec \mathscr{P}_2 \tag{5.2}$$

The cancellations in $\sum_{\mathscr{P}}$ will arise after a suitable rearrangement:

Lemma 5.2: For any function $h(\mathscr{P})$ of neutral partitions, fixed hierarchy $\mathscr{H}$, and neutral hierarchies $\mathscr{M} \subset \mathscr{N} \subset \mathscr{H}$,

$$\sum_{\{\mathscr{P} | \nu(\mathscr{P}) = \mathscr{N}\}} h(\mathscr{P}) = \sum_{\substack{\mathscr{Q} \prec \pi(\mathscr{M}) \\ \nu(\mathscr{Q}) = \mathscr{N}}} \; \sum_{\{\mathscr{P} | \mathscr{P} \wedge \pi(\mathscr{M}) = \mathscr{Q}\}} h(\mathscr{P})$$

Proof. We claim that, for any $\mathscr{M}$ and $\mathscr{P}$, (i) $\mathscr{P} \wedge \pi(\mathscr{M})$ exists if and only if $\mathscr{M} \subset \nu(\mathscr{P})$, and (ii) in this case, $\nu(\mathscr{P}) = \nu(\mathscr{P} \wedge \pi(\mathscr{M}))$. From this the lemma follows, since for any $\mathscr{P}$ with $\nu(\mathscr{P}) = \mathscr{N}$ we may set $\mathscr{Q} = \mathscr{P} \wedge \pi(\mathscr{M})$ to show that $\mathscr{P}$ occurs on the right-hand side; conversely, given $\mathscr{Q}$, $\mathscr{P}$ from this side, $\nu(\mathscr{P}) = \nu(\mathscr{Q}) = \mathscr{N}$ and moreover $\mathscr{Q} = \mathscr{P} \wedge \pi(\mathscr{M})$ is determined by $\mathscr{P}$, so that $\mathscr{P}$ occurs only once.

We check the claim: (i) $\mathscr{P} \wedge \pi(\mathscr{M})$ exists iff $e(P \cap (A/\mathscr{M})) = 0$ for all $A \in \mathscr{M}$ and $P \in \mathscr{P}$ iff $e(P \cap A) = 0$ for all such A and P iff $\mathscr{M} \subset \nu(\mathscr{P})$. (ii) $\nu(\mathscr{P} \wedge \pi(\mathscr{M})) \subset \nu(\mathscr{P})$ by (5.2); the opposite inclusion will follow from (i) once we show that $((\mathscr{P} \wedge \pi(\mathscr{M})) \wedge \pi(\nu(\mathscr{P}))$ exists. But (5.1) implies that $\pi(\nu(\mathscr{P})) \prec \pi(\mathscr{M})$, so that $\mathscr{P} \wedge \pi(\nu(\mathscr{P}))$ [which exists by (i)] is a common refinement of $\mathscr{P}$, $\pi(\mathscr{M})$, and $\pi(\nu(\mathscr{P}))$. ∎

This rearrangement is motivated by the following:

Lemma 5.3. For neutral partitions $\mathscr{Q}_1, \mathscr{Q}_2$ with $\mathscr{Q}_1 \prec \mathscr{Q}_2$ and $\mathscr{Q}_2 \neq \{I\}$,

$$\sum_{\{\mathscr{P} | \mathscr{P} \wedge \mathscr{Q}_2 = \mathscr{Q}_1\}} (-1)^{|\mathscr{P}|-1} (|\mathscr{P}| - 1)! = 0 \tag{5.3}$$

Proof. This is standard combinatorics. Thus by Ref. 6, Corollary to Proposition 3, Section 8, $\{\mathscr{P} | \mathscr{P} \succ \mathscr{Q}_1\}$ is a lattice with Möbius function

$$\mu(\mathscr{P}, \{I\}) = (-1)^{|\mathscr{P}|-1} (|\mathscr{P}| - 1)!$$

(5.3) then follows by Ref. 6, Corollary to Proposition 4, Section 5. ∎

We next introduce scaling variables for neutral clusters. Fix a hierarchy $\mathscr{H}$, a point $x \in X_{\mathscr{H}}$, and a neutral hierarchy $\mathscr{N} \subset \mathscr{H}$, introduce real variables $(t_A)_{A\in\mathcal{N}}$ satisfying $|t_A|\leqslant 1$, and set $\tau_A=\prod_{A\subset C\in\mathcal{N}} t_C$, $\tau_{AB}=\tau_A\tau_B^{-1}$ for $A, B\in\mathcal{N}$, $A\subset B$. Then we define $y(x, t)\in\mathbb{R}^{2N}$ inductively by

$$\begin{aligned} &y_1(x, t)=x_1 \\ &y_{i_B}(x, t)=y_{i_A}(x, t)+\tau_A(x_{i_B}-x_{i_A}) \qquad \text{for} \quad A\in\mathcal{N}, B\in\mathcal{N}_1(A) \end{aligned} \tag{5.4}$$

We collect simple properties of these new variables in the following:

Remark 5.4. (a) $y(x, t)|_{t_A\equiv 1}=x$.

(b) $r_A(\mathcal{H}; y)=\tau_B r_A(\mathcal{H}; x)$ [respectively $R_A(\mathcal{H}; y)=\tau_C R_A(\mathcal{H}; x)$], where $A\in\mathcal{H}$ and $B\in\mathcal{N}$ (respectively $C\in\mathcal{N}$) is the minimal element containing A (respectively A^*). Thus $y(x, t)\in X_{\mathcal{H}}$.

(c) For $i, j\in I$ let $A_{ij}\equiv A$ be the minimal element $\mathcal{N}$ containing i and j, and suppose $i\in B\in\mathcal{N}_0(A)$, $j\in C\in\mathcal{N}_0(A)$. Then

$$y_i-y_j=\tau_A z_{ij}$$

with

$$\begin{aligned} z_{ij}(x, t)=x_{i_B}+&\sum_{\substack{D\ni i\\ D\subset B, D\neq B}}\tau_{D^*A}(x_{i_D}-x_{i_{D^*}})-x_{i_C} \\ &-\sum_{\substack{D\ni j\\ D\subset C, D\neq C}}\tau_{D^*A}(x_{i_D}-x_{i_{D^*}}) \end{aligned} \tag{5.5}$$

Here $D\in\bar{\mathcal{N}}$ and D^* is calculated in $\mathcal{N}$. Moreover, from $y\in X_{\mathcal{H}}$ and Lemmas 3.3(d) and 3.5,

$$\begin{aligned} |z_{ij}(x, t)| &\geqslant (1-3\alpha)(1-\alpha)^{-1}\,|x_{i_B}-x_{i_C}| \\ &\geqslant (1-3\alpha)(1-\alpha)^{-1}\left(\frac{\alpha}{1+\alpha}\right)^{N-2} r_A(\mathcal{H}; x) \end{aligned} \tag{5.6}$$

Lemma 5.5. For $\mathcal{H}$ and $x\in X_{\mathcal{H}}$ as above, a neutral partition $\mathcal{P}$, and $\mathcal{N}=\nu(\mathcal{P})$, define

$$\hat{f}_{\mathcal{P}}(\beta, x, t)=\prod_{A\in\mathcal{N}}|t_A|^{\beta/2\sum_{i\in A}e_i^2} f_{\mathcal{P}}(\beta, y(x, t)) \tag{5.7}$$

Then when all x_i are distinct $\hat{f}$ is smooth in t (for $|t_A|\leqslant 1$, all $A\in\mathcal{N}$) and, if $\mathcal{M}\subset\mathcal{N}$,

$$\hat{f}_{\mathcal{P}}(\beta, x, t)|_{t_A=0, A\in\mathcal{M}}=\hat{f}_{\mathcal{P}\wedge\pi(\mathcal{M})}(\beta, x, t) \tag{5.8}$$

Proof. From (2.5) we have immediately

$$\hat{f}_{\mathscr{P}} = \prod_{P \in \mathscr{P}} \prod_{\substack{i,j \in P \\ i<j}} |z_{ij}(x, t)|^{\beta e_i e_j} \tag{5.9}$$

and the smoothness follows from (5.5) and (5.6). Now let us use a superscript to denote the set of scaling variables, so that (5.4) defines $y^{\mathscr{N}}(x, t)$ and $y^{\mathscr{N}}(x, t) = y^{\mathscr{M}}(y^{\mathscr{N} \setminus \mathscr{M}}(x, t), t)$. Then from (5.5),

$$\hat{f}\,|_{t_A = 0, A \in \mathscr{M}} = \prod_{D \in \mathscr{N} \setminus \mathscr{M}} |t_D|^{(\beta/2)\Sigma_{i \in D} e_i^2} \times \prod_{P \in \mathscr{P}} \prod_{A \in \mathscr{M}} \prod_{\substack{C, D \in \mathscr{M}_0(A) \\ C \neq D}} |y_{i_C}^{\mathscr{N} \setminus \mathscr{M}} - y_{i_D}^{\mathscr{N} \setminus \mathscr{M}}|^{(\beta/2) e(P \cap C) e(P \cap D)} \tag{5.10}$$

Since $\mathscr{M} \subset \mathscr{N} = v(\mathscr{P})$, $e(P \cap C) = 0$ unless $C \in \bar{\mathscr{M}} \setminus \mathscr{M}$ (i.e., $|C| = 1$) and similarly for D. Thus (5.10) is precisely $\hat{f}_{\mathscr{P} \wedge \pi(\mathscr{M})}(\beta, x, t)$. ∎

We can now give the following.

Proof of Lemma 4.3. We must estimate

$$\left(\prod_{A \in \mathscr{N}(\mathscr{H}) \setminus \mathscr{A}} K_A \right) F_{\mathbf{N}} = \sum_{\mathscr{N}} \sum_{\{\mathscr{P} | v(\mathscr{P}) = \mathscr{N}\}} \left(\left(\prod K_A \right) f_{\mathscr{P}} \right) \psi_{\mathscr{P}} (-1)^{|\mathscr{P}| - 1} (|\mathscr{P}| - 1)! \tag{5.11}$$

where we have used Lemma 3.6. In (5.11) we write $f_{\mathscr{P}} = \hat{f}_{\mathscr{P}}|_{t_A \equiv 1}$ and

$$\hat{f}_{\mathscr{P}} = \prod_{A \in \mathscr{N} \setminus \{I\}} [T_A + (1 - T_A)]\, \hat{f}_{\mathscr{P}} \tag{5.12}$$

where T_A extracts the Taylor series in t_A up to order $n_A - 1$ (and $T_A = 0$ if $n_A = 0$). We claim that when (5.12) is expanded and inserted into (5.11), only the term

$$\prod_{A \in \mathscr{N} \setminus \{I\}} (1 - T_A)\, \hat{f}_{\mathscr{P}}\,|_{t_A \equiv 1}$$

will survive. For any first-order term from a T_A (with $n_A = 2$) will be annihilated by K_A [use $\hat{f}_{\mathscr{P}}(S_A(\theta + \pi)\, x, t) = \hat{f}_{\mathscr{P}}(x, t')$, with $t'_B = (-1)^{\delta_{AB}}\, t_B$]. Any remaining term is by Lemma 5.5 of the form

$$\prod_{A \in \mathscr{N} \setminus \mathscr{M}} (1 - T_A)\, \hat{f}_{\mathscr{P} \wedge \pi(\mathscr{M})}\,|_{t_A \equiv 1}$$

for some $\mathscr{M} \subset \mathscr{N}$ with $\mathscr{M} \supsetneqq \{I\}$. When we sum these terms over $\mathscr{P}$ with $v(\mathscr{P}) = \mathscr{N}$ (for fixed $\mathscr{M}$), rearrange the sum according to Lemma 5.2, and use Lemma 5.3 and the easily derived formula $\psi_{\mathscr{P}}(x) = \psi_{\mathscr{P} \wedge \pi(\mathscr{M})}(x)$ for $x \in X_{\mathscr{H}}$, the result vanishes unless $\mathscr{M} = \{I\}$ [corresponding to $\mathscr{Q}_2 = \pi(\mathscr{M}) = \{I\}$ in Lemma 5.3].

Thus we have the decomposition (4.9) with

$$g_{\mathscr{P},\mathscr{H},\mathscr{A}} = (-1)^{|\mathscr{P}|-1}(|\mathscr{P}|-1)!\,\psi_{\mathscr{P}}(x) \times \left(\prod_{A \in \mathscr{N}(\mathscr{H}) \setminus \mathscr{A}} K_A \right) \prod_{A \in v(\mathscr{P}) \setminus \{I\}} (1 - T_A) \hat{f}_{\mathscr{P}} |_{t_A \equiv 1} \tag{5.13}$$

We write

$$\prod_{A \in v(\mathscr{P}) \setminus \{I\}} (1 - T_A) \hat{f}_{\mathscr{P}} |_{t_A \equiv 1} = \prod_{A \in v(\mathscr{P})} \left(\int_0^1 (1 - t_A)^{n_A - 1} \, dt_A \right) (D_n \hat{f})(\beta, x, t)$$

where $D_n \equiv \prod_{A \in v(\mathscr{P})} (\partial/\partial t_A)^{n_A}$; then (4.10) will follow from (4.2) and the estimate

$$|(D_n \hat{f}_{\mathscr{P}})(\beta, x, t)| \leqslant C \prod_{A \in \mathscr{N}} \left(\frac{r_A(\mathscr{H}; x)}{r_{A^*}(\mathscr{H}; x)} \right)^{n_A} \hat{f}_{\mathscr{P}}(\beta, x, t) \tag{5.14}$$

for all $|t_A| \leqslant 1$. To verify (5.14), note that from (5.9)

$$D_n \hat{f}_{\mathscr{P}} = \sum_{k \geqslant 1} \sum_{m,\gamma} c_{m,\gamma} \prod_{l=1}^{k} \left(\frac{D_{m_{l1}} z_{\gamma_l} \cdot D_{m_{l2}} z_{\gamma_l}}{|z_{\gamma_l}|^2} \right) \hat{f}_{\mathscr{P}} \tag{5.15}$$

Here $m = (m_{11}, m_{12}, m_{21}, \ldots, m_{k1}, m_{k2})$ with m_{lr} a multi-index $(m_{lrA})_{A \in v(\mathscr{P})}$, $m_{l1} \neq 0$ for all l, and $\sum_{l,r} m_{lr} = n$, and $\gamma = (\gamma_1, \ldots, \gamma_k)$ with $\gamma_l \in \{(i, j) \mid i, j \in P \in \mathscr{P}, i < j\}$. From (5.5) we see that $D_{m_{lr}} z_{ij}$ is nonzero only if

$$m_{lrA} = \sum_{p=1}^{q} \delta_{AC_p}$$

for some chain $C_1 \subsetneqq C_2 \subsetneqq \cdots \subsetneqq C_q$ of sets in $\mathscr{N}$, satisfying $C_q \subsetneqq A_{ij}$ [notation of Remark 5.4(c)] and $i \in C_1$ or $j \in C_1$. In this case, from (5.5) and (5.6),

$$\frac{|D_{m_{lr}} z_{ij}|}{|z_{ij}|} \leqslant C \frac{r_{C_1}(\mathscr{H}; x)}{r_A(\mathscr{H}; x)} \leqslant C \prod_{p=1}^{q} \frac{r_{C_p}(\mathscr{H}; x)}{r_{C_p^*}(\mathscr{H}; x)} \tag{5.16}$$

and (5.14) follows from (5.15) and (5.16). ∎

6. ESTIMATING THE INTEGRALS

Throughout this section we fix a hierarchy $\mathscr{H}$, an ancestral subfamily $\mathscr{A} \subset \mathscr{H}$, and a neutral partition $\mathscr{P}$; we usually omit these arguments, e.g., for $A \in \bar{\mathscr{H}}$, $\mu_A \equiv \mu_A(\mathscr{P}, \mathscr{A})$, etc. We define $\lambda_A \equiv \mu_A(\mathscr{P}, \varnothing)$ and

$$\tilde{\mu}_A \equiv \inf_{\mathscr{D}} \sum_{B \in \mathscr{D}} \mu_B \tag{6.1}$$

(similarly for $\tilde{\lambda}_A$), where the infimum is over all subfamilies $\mathscr{D} \subset \bar{\mathscr{H}}$ which partition A. Note that $\tilde{\mu}_A \leqslant 0$ since $\mu_{\{i\}} \equiv 0$.

We will prove Lemma 4.4 from an inductive bound on the integral taken over variables associated with clusters of increasing size; thus for each $A \in \mathscr{H}$ we will integrate over configurations of the centers of maximal subclusters in A which (i) yield a cluster A of some maximal size r, (ii) satisfy hard core restriction, (iii) for $A \in \mathscr{A}$, satisfy boundary restrictions, (iv) for maximal subclusters which lie in $\mathscr{A}$, are consistent with hard core and boundary restrictions on those subclusters, and (v) for $\mathscr{A} \neq \varnothing$, lie in Λ (recall that in Lemma 4.4 we integrate over $X_{\mathscr{H},\mathscr{A}} \subset \Lambda$ except when $\mathscr{A} = \varnothing$). We formalize this in the following.

Definition 6.1. (a) For $A \in \mathscr{H}$ define $Z_A(x_{i_A}, r)$ to be the set of $|\mathscr{H}_1(A)|$-tuples $(x_{i_B})_{B \in \mathscr{H}_1(A)}$ such that (i) $r_A \equiv \sup_B |x_{i_A} - x_{i_B}| \leqslant r$, (ii) $r_A \geqslant \delta_A$, (iii) $r_A \geqslant (1-\alpha)\, s_A$, if $A \in \mathscr{A}$, (iv) for $B \neq A_*$, $|x_{i_B} - x_{i_A}| \geqslant \alpha^{-1} \delta_B$, and if also $B \in \mathscr{A}$, $|x_{i_B} - x_{i_A}| \geqslant \alpha^{-1}(1-\alpha)\, s_B$, and (v) $x_{i_B} \in \Lambda$, for all $B \in \mathscr{H}_1(A)$, if $\mathscr{A} \neq \varnothing$.

(b) Define inductively:

$$J_{\{i\}}(x_i, r) = 1$$

and for $A \in \mathscr{H}$,

$$J_A(x_{i_A}, r) = \int_{Z_A(x_{i_A}, r)} r_A^{\mu_A - \sum_A \mu_B - 2|\mathscr{H}_1(A)|} \prod_{B \in \mathscr{H}_0(A)} J_B(x_{i_B}, \alpha r_A) \prod_{B \in \mathscr{H}_1(A)} dx_{i_B} \tag{6.2}$$

Now our inductive estimate is as follows.

Lemma 6.2. For $A \in \mathscr{H}$,

$$J_A(x_{i_A}, r) \leqslant C t_A^{\eta_A} r^{\gamma_A}\, d^{\tilde{\lambda}_A} [1 + \log(r/d)]^{m_A} \tag{6.3}$$

where $t_A \equiv \max(s_A, \delta_A)$, with constants η_A and γ_A satisfying

$$\begin{aligned} \eta_A &= 0, && \text{if} \quad A \notin \mathscr{A} \quad \text{or} \quad \delta_A = 0 \\ 0 \geqslant \eta_A &\geqslant -1, && \text{if} \quad A \in \mathscr{A} \end{aligned}$$

and

$$\eta_A + \gamma_A = \mu_A - \tilde{\lambda}_A$$

The logarithmic power may be bounded by

$$m_A \leqslant |\{B \in \mathscr{H} \mid B \subsetneqq A\}| + |\{B \text{ minimal in} \mathscr{H} \mid B \subset A\}|$$

We have stated Lemma 6.2 in full generality, but for clarity, but for clarity we will first prove it in the special case $\mathscr{A} = \varnothing$, for which boundary considerations are absent, and from this derive Lemma 4.4(a). We will then consider boundary effects.

Proof of Lemma 6.2. Case 1, $\mathscr{A} = \varnothing$. In this case $\mu_B = \lambda_B$ for all $B \subset A$ and (6.3) reduces to

$$J_A(x_{i_A}, r) \leqslant C r^{\mu_A - \tilde{\mu}_A} d^{\tilde{\mu}_A} [1 + \log(r/d)]^{m_A} \tag{6.4}$$

We argue inductively by inserting the bound (6.4) for each J_B into (6.2), then making the variable change (4.5a) and integrating over all ξ_{AB} satisfying (4.5b). This yields

$$J_A \leqslant C d^{\sum_A \tilde{\mu}_B} [1 + \log(r/d)]^{\sum_A m_B} \int_{\delta_A}^{r} r_A^{\mu_A - \sum_A \tilde{\mu}_B} \frac{dr_A}{r_A} \tag{6.5}$$

Then (6.4) follows by separate consideration of the cases (i) $\mu_A > \sum_A \tilde{\mu}_B$, where $\tilde{\mu}_A = \sum_A \tilde{\mu}_B$, and (ii) $\mu_A \leqslant \sum_A \tilde{\mu}_B$, where $\tilde{\mu}_A = \mu_A$ and $\delta_A = d$. We remark that in this case the power m_A on the logarithm can be inductively bounded by $|\{B \in \mathscr{H} \mid B \subset A, \tilde{\mu}_B = \mu_B\}|$. ∎

Proof of Lemma 4.4(a). Take $\mathscr{A} = \varnothing$, so $\mu_A \equiv \lambda_A$. Then

$$\int_{\substack{X_{\mathscr{H}}(x_1) \\ r_I \geqslant u}} G_{\mathscr{P}, \mathscr{H}, \varnothing}\, dx_2 \cdots dx_N \leqslant \int_{r_I \geqslant u} r_I^{\mu_I - \sum_I \mu_A - 2|\mathscr{H}_1(I)|} \times \prod_{A \in \mathscr{H}_0(I)} J_A(x_{i_A}, \alpha r_I) \prod_{A \in \mathscr{H}_1(I)} dx_{i_A}$$

The inductive estimate (6.4) of J_A and the same change of variables as in the proof of Lemma 6.2 lead to

$$\int G_{\mathscr{P}, \mathscr{H}, \varnothing}\, dx_2 \cdots dx_N \leqslant C\, d^{\sum_I \tilde{\mu}_A} \int_u^{\infty} r_I^{\mu_I - \sum_I \tilde{\mu}_A} [1 + \log(r_I/d)]^{\sum_I m_A} \frac{dr_I}{r_I}$$

and it suffices to show that

$$\mu_I - \sum_I \tilde{\mu}_A \leqslant -\varepsilon \tag{6.6}$$

But for some partition $\mathscr{D}$ of I with $\mathscr{D} \neq I$,

$$\begin{aligned}\mu_I - \sum_I \tilde{\mu}_A &= \mu_I - \sum_{A \in \mathscr{D}} \mu_A \\ &= \sum_{A \in \mathscr{D}} \left[2 - 2\chi_{\nu(\mathscr{P})}(A) - \frac{\beta}{2} \sum_{P \in \mathscr{P}} e(P \cap A)^2 \right] - 2 \\ &\leqslant \sum_{\substack{B \in \mathscr{D}' \\ e(B) \neq 0}} \left[2 - \frac{\beta}{2} e(B)^2 \right] - 2 \\ &\leqslant -\varepsilon\end{aligned}$$

by (2.8), where $\mathscr{D}' = \{A \cap P \mid A \in \mathscr{D}, P \in \mathscr{P}\}$. ▌

We now turn to the general case of Lemma 6.2. We need one additional estimate.

Lemma 6.3. Let Λ_ρ be the disk $|x| \leqslant \rho$ and for $x \in \Lambda_\rho$ write $t(x) = \max\{d, \rho - |x|\}$. For $x_0 \in \Lambda_\rho$ and $\sigma > 0$ define $\Lambda_\rho(x_0, \sigma) = \{x \in \Lambda_\rho \mid \sigma \geqslant |x - x_0| \geqslant \alpha^{-1}(1-\alpha)\, t(x)\}$. Then if η and γ are real with $0 \geqslant \eta \geqslant -1$, we have

$$\int_{\Lambda_\rho(x_0,\sigma)} |x - x_0|^\gamma\, t(x)^\eta\, d^2x$$

$$\leqslant \begin{cases} C\sigma^{\gamma+\eta+2}[1 + \log(\sigma/d)]^m, & \text{if} \quad \gamma+\eta+2 \geqslant 0 \\ Ct(x_0)^{\gamma+\eta+2}[1 + \log(\sigma/d)]^m, & \text{if} \quad \gamma+\eta+2 < 0 \end{cases}$$

where $m = \delta_{\eta,-1} + \delta_{\gamma+\eta+2,0}$, and C may depend on α, η, γ.

Proof. We sketch a proof which, while not completely avoiding the consideration of various special cases, at least reduces geometric complications. Write $f \sim g$ when $cf \leqslant g \leqslant c^{-1} f$ for some positive c (possibly α, η, and γ dependent). We assume that $\rho \gg d$, since the case $\rho \lesssim d$ is trivial, and without loss of generality that $\sigma \leqslant 2\rho$.

Let $r = |x|$, $a = |x_0|$, and $w = |x - x_0|$, and introduce new integration variables $u = \rho - r$, $v = w + a - r$. The inverse mapping $(u, v) \mapsto x$ is 2–1 (or 0–1) almost everywhere and

$$d^2x = 2rw[(r + a + w)(r + a - w)(r + w - a)(w + a - r)]^{-1/2}\, du\, dv$$

Simple algebra shows that, in the integration region,

$$0 \leqslant u \leqslant \alpha(v+\rho-a)$$

$$(1-\alpha)(v+\rho-a) \leqslant w \leqslant (v+\rho-a)$$

$$r+a+w \geqslant r$$

$$r+w-a \geqslant (1-2\alpha)(v+\rho-a)$$

and

$$v_1 \leqslant v \leqslant v_2$$

where $v_1 = \max\{0, d+a-\rho\}$ and $v_2 = \min\{2a, (1-\alpha)^{-1}\sigma+a-\rho\}$. Thus the integral is bounded by

$$C\int_{v_1}^{v_2} dv \int_0^{\alpha(v+\rho-a)} du\, u^{\eta}(v+\rho-a)^{1/2+\gamma}\left[\frac{\rho}{v(2a-v)}\right]^{1/2}$$

$$\leqslant C'[1+\log(\sigma/d)]^{\delta_{\eta,-1}}\int_{v_1}^{v_2} dv(v+\rho-a)^{3/2+\gamma+\eta}\left[\frac{\rho}{v(2a-v)}\right]^{1/2} \qquad (6.7)$$

Now if $a \leqslant \frac{1}{2}\rho$ the integral (6.7) vanishes unless $\sigma \sim \rho$ (otherwise $v_2 < 0$) and the bound $C\rho^{2-\gamma-\eta}$ $(\sim C\sigma^{2-\gamma-\eta} \sim Ct(x_0)^{2-\gamma-\eta})$ follows from $(v+\rho-a) \sim \rho$. For $a \geqslant \frac{1}{2}\rho$ we write

$$\int_{v_1}^{v_2} \cdots dv = \int_{[v_1,v_2]\cap[0,t(x_0)]} \cdots dv + \int_{[v_1,v_2]\cap[t(x_0),a]} \cdots dv + \int_{[v_1,v_2]\cap[a,2a]} \cdots dv$$

The first term is bounded by $Ct(x_0)^{2+\gamma+\eta}$ [use $(v+\rho-a) \sim t$ and $(2a-v) \sim \rho$], the third by $C\sigma^{2+\gamma+\eta}[(v+\rho-a) \sim v \sim \rho]$, and the second by the maximum of these $[v \sim (v+\rho-a)$ and $(2a-v) \sim \rho]$. ∎

Proof of Lemma 6.2. General case. By the proof of case 1, (6.3) is valid for $A \notin \mathscr{A}$. For $A \in \mathscr{A}$, we estimate inductively as in the proof of the special case, but now the substitution (4.5) is not appropriate, and we have

$$J_A \leqslant Cd^{\Sigma_A \lambda_B}[1+\log(r/d)]^{\Sigma_A m_B}$$

$$\times \int_{Z_A(x_{i_A}, r)} r_A^{\mu_A - \Sigma_A(\eta_B+\lambda_B) - 2|\mathscr{H}_1(A)|} \prod_{B\in\mathscr{H}_0(A)} t_B^{\eta_B} \prod_{B\in\mathscr{H}_1(A)} dx_{i_B} \qquad (6.8)$$

The integration region may be decomposed as $Z_A = \bigcup_{C\in\mathscr{H}_1(A)} Z_{AC}$, where $Z_{AC} = \{x \in Z_A \mid r_A = |x_{i_c} - x_{i_A}|\}$. In Z_{AC} we estimate integrals over each variable x_{i_B}, $B \in \mathscr{H}_1(A)$: (i) For $B \neq C$ we must by Definition 6.1 integrate $t_B^{\eta_B}\, dx_{i_B}$ over either the region $r_A \geqslant |x_{i_B} - x_{i_A}| \geqslant \alpha^{-1}\delta_B$ or, if $B \in \mathscr{A}$, over

$\Lambda_\rho(x_{i_A}, r_A)$; by the induction hypothesis $\eta_B \neq 0$ is possible only in the second case. By direct calculation, or by Lemma 6.3 in the second case, the integral is bounded by

$$r_A^{2+\eta_B}[1+\log(r_A/d)] \tag{6.9}$$

(ii) Finally we integrate

$$t_{A_*}^{\eta_{A_*}} r_A^{\mu_A - \sum_A \bar{\lambda}_B - \eta_C - \eta_{A_*} - 2} t_C^{\eta_C}\, dx_{i_C}$$

over either $r \geqslant |x_{i_C} - x_{i_A}| \geqslant t_A$ or $\Lambda_\rho(x_{i_A}, r)$, again $\eta_C \neq 0$ is possible only in the second case. As above, and from $t_{A_*} = t_A$, the integral is bounded by

$$\begin{cases} C t_A^{\eta_{A_*}} r^{\mu_A - \sum_A \bar{\lambda}_B - \eta_{A_*}}[1+\log(r/d)]^b, & \text{if } \mu_A \geqslant \sum_A \bar{\lambda}_B - \eta_{A_*} \\ C t_A^{\mu_A - \sum_A \bar{\lambda}_B}[1+\log(r/d)]^b, & \text{if } \mu_A \leqslant \sum_A \bar{\lambda}_B - \eta_{A_*} \end{cases} \tag{6.10}$$

We can now verify (6.3). Logarithms are collected from (6.8), (6.9), and (6.10); the inductive bound on m_A follows since $b \leqslant 2$ and $b \leqslant 1$ if A is minimal in $\mathscr{H}$. To obtain η_A, γ_A, and $\bar{\lambda}_A$ from (6.10) we consider special cases. If $\mu_A = \lambda_A$ [i.e., $A \notin \mathscr{A} \cap v(\mathscr{P})$ or $A = I$] the three cases $\lambda_A > \sum_A \bar{\lambda}_B$, $\sum_A \bar{\lambda}_B \geqslant \lambda_A \geqslant \sum_A \bar{\lambda}_B - \eta_{A_*}$, and $\sum_A \bar{\lambda}_B - \eta_{A_*} > \lambda_A$ lead to (6.3) with $\eta_A = \eta_{A_*}$, $\eta_A = \eta_{A_*}$, and $\eta_A = 0$ (and $\bar{\lambda}_A = \sum_A \bar{\lambda}_B$, $\bar{\lambda}_A = \lambda_A$, $\bar{\lambda}_A = \lambda_A$), respectively. Similarly, if $\mu_A = \lambda_A - 1$ [i.e., $A \in \mathscr{A} \cap v(\mathscr{P})$ and $A \neq I$] the cases $\lambda_A > \sum_A \bar{\lambda}_B + 1 - \eta_{A_*}$, $\sum_A \bar{\lambda}_B + 1 - \eta_{A_*} \geqslant \lambda_A \geqslant \sum_A \bar{\lambda}_B$, and $\sum_A \bar{\lambda}_B > \lambda_A$ lead to (6.3) with $\eta_A = \eta_{A_*}$, $\eta_A = \mu_A - \bar{\lambda}_A$, and $\eta_A = -1$ (and $\bar{\lambda}_A = \sum_A \bar{\lambda}_B$, $\bar{\lambda}_A = \sum_A \bar{\lambda}_B$, $\bar{\lambda}_A = \lambda_A$), respectively. ∎

Finally we give the following.

Proof of Lemma 4.4(b). From Lemmas 6.2 and 6.3,

$$\begin{aligned} \int_{X_{\mathscr{H},\mathscr{A}}} G_{\mathscr{P},\mathscr{H},\mathscr{A}}\, dx_1 \cdots dx_N &= \int_{\Lambda_\rho} J_I(x_1, 2\rho)\, dx_1 \\ &\leqslant C(2\rho)^{\gamma_I} d^{\bar{\lambda}_I}[1+\log(2\rho/d)]^{m_I} \int_\Lambda t_I^{\eta_I}\, dx_1 \\ &\leqslant C d^{\bar{\lambda}_I} \rho^{\lambda_I - \bar{\lambda}_I}[1+\log(2\rho/d)]^{m_I+1} \end{aligned}$$

since $\eta_I + \gamma_I = \mu_I - \bar{\lambda}_I = \lambda_I - \bar{\lambda}_I$. But from (6.6), $\bar{\lambda}_I = \sum_I \bar{\lambda}_A$ and thus $\lambda_I - \bar{\lambda}_I \leqslant -\varepsilon$, which implies the bound of Lemma 4.4(b). ∎

7. CORRELATION FUNCTIONS

We sketch briefly the application of the preceding methods to the coefficients of the Mayer series of correlation functions. Let $\mathbf{n} = (n^1,..., n^s)$

denote a collection of particles, with $n = \sum n^s > 0$; for $x = (x_1, ..., x_n) \in \mathbb{R}^{2n}$ the truncated correlation function for n particles in the volume Λ, with species given by $\mathbf{n}$ and positions by x, is easily seen to have the expansion [compare (2.3)]

$$\rho_n^T(\beta, \mathbf{z}, x; \Lambda) = \sum_{\substack{\mathbf{N} \geqslant \mathbf{n} \\ \mathbf{N} \cdot \mathbf{e} = 0}} \frac{\mathbf{z}^{\mathbf{N} - \mathbf{n}}}{(\mathbf{N} - \mathbf{n})!} \int_{\Lambda^{N-n}} F_{\mathbf{N}}(\beta, x)\, dx_{n+1} \cdots dx_N \tag{7.1}$$

The infinite volume limit of the integrals in (7.1) may be controlled by the techniques of Sections 3–6.

Specifically, we define regions $X_{\mathscr{H}, \mathscr{A}} \in \mathbb{R}^{2N}$ as in Section 3, then estimate the integral with respect to $x_{n+1}, ..., x_N$ over the section $X_{\mathscr{H}, \mathscr{A}}(x_1, ..., x_n)$ (which will be empty unless $\mathscr{H}$ is compatible with the clustering inherent in x). The only difficulty is that we cannot average over the orientations of clusters containing more than one particle from $I_0 \equiv \{1, ..., n\}$. Instead, we achieve the needed falloff between neutral clusters containing I_0 and disjoint clusters as follows:

(i) In the equivalent of Lemma 4.3 we omit operators K_A for $|A \cap I_0| \geqslant 2$, but for each $A \in \mathscr{N}(\mathscr{H})$ with $I_0 \subset A$ we include an operator K'_A averaging over rotations about x_1 of all particles not in A:

$$(K'_A f)(x) = (2\pi)^{-1} \int_0^{2\pi} f[S_A(\theta)\, S_I(-\theta)\, x]\, d\theta$$

(ii) We consider only volumes $\Lambda = \Lambda_{1a}$ which are disks of radius a centered at x_1, so that the rotations K'_A do not interact with the boundary.

The proof of the existence of the thermodynamic limit for the coefficients in (7.1) now proceeds as for those of the pressure. The condition for the existence of the limit is [compare (2.8)]

$$\varepsilon_1 \equiv 2 - \sup_{\mathscr{D}} \sum_{\substack{A \in \mathscr{D} \\ e(A) \neq 0}} \left[2 - \frac{\beta}{2} e(A)^2 \right] > 0 \tag{7.2}$$

where now the supremum is over partitions $\mathscr{D}$ of I for which $I_0 \subset D$ for some $D \in \mathscr{D}$.

Remark 7.1. The definition of Λ_{1a} could easily be replaced by the requirement that the center of the disk stay within a bounded distance of x_1 during the thermodynamic limit. More general limits in which, say, $(a - |x_1|) \sim a^{\xi}$ for $\xi < 1$ could also be treated by omitting the averaging operators K'_A and strengthening the condition (7.2).

The estimates developed in Sections 5 and 6 will here also yield information on the asymptotic behavior of correlation functions. We illustrate with the case $n=2$.

Theorem 7.2. Fix x_1, x_2 and $\mathbf{n}=(n^1,..., n^s)$ with $n=2$. Then

$$\rho^T_{\mathbf{n},\mathbf{N}}(\beta, x_1, x_2) \equiv \lim_{a\to\infty} \int_{A_{1a}^{N-2}} F_{\mathbf{N}}(\beta, x)\, dx_3 \cdots dx_N$$

satisfies

$$|\rho^T_{\mathbf{n},\mathbf{N}}(\beta, x_1, x_2)| \leqslant C\,|x_1-x_2|^{-\zeta}[1+\log(|x_1-x_2|/d)]^{m_C} \tag{7.3}$$

with m_C as in Lemma 6.2 and with

$$\zeta = 4 - \sup_{\mathscr{D}} \sum_{\substack{A\in\mathscr{D}\\ e(A)\neq 0}} \left[2-\frac{\beta}{2}e(A)^2\right] \tag{7.4}$$

In (7.4), in contrast to (7.2), $\mathscr{D}$ runs over partitions of I which split I_0: $1\in D$, $2\in D'$, with $D, D'\in\mathscr{D}$ and $D\neq D'$.

Proof sketch. Fix a hierarchy $\mathscr{H}$, and let $C\in\mathscr{H}$ be the minimal element with $I_0\subset C$. The key quantities in the proofs of Sections 4–6 are $g_{\mathscr{P},\mathscr{H},\mathscr{A}}$ and J_A [see (5.13) and (6.2)]; we must redefine them, and for simplicity consider only $\mathscr{A}=\varnothing$, since terms with $\mathscr{A}\neq\varnothing$ vanish in the thermodynamic limit. $g_{\mathscr{P},\mathscr{H},\mathscr{A}}$ is replaced by $g'_{\mathscr{P},\mathscr{H},\mathscr{A}}$, defined as in (5.13) but with

$$\prod_{A\in\mathscr{N}(\mathscr{H})} K_A$$

replaced by

$$\prod_{\substack{A\in\mathscr{N}(\mathscr{H})\\ A\not\supset C}} K_A \prod_{\substack{A\in\mathscr{N}(\mathscr{H})\\ A\supset C}} K'_A$$

Because $|I_0|=n=2$ we are averaging over orientations for all neutral sets and the estimate of Lemma 4.3 is unchanged.

For $|A\cap I_0|<2$, $J'_A(x_{i_A}, r)\equiv J_A(x_{i_A}, r)$ is defined by (6.2). For $A=C$ we do not integrate over x_2 and, moreover, we must have

$$b_1\,|x_1-x_2| \leqslant r_C \leqslant b_2\,|x_1-x_2| \tag{7.5}$$

for some constants b_1, b_2, by Lemmas 3.3 and 3.5. Thus we define

$$J'_C(x_1, x_2) = \int_{Z'_C(x_{i_C}, b_1|x_1-x_2|)} r_C^{\mu_C - \Sigma_A \mu_B - 2|\mathscr{H}_1(C)|} \prod_{B\in\mathscr{H}_0(C)} J_B(x_{i_B}, \alpha r_C) \prod_{\substack{B\in\mathscr{H}(C)\\ B\not\ni 2}} dx_{i_B}$$

with Z'_C the set of $(|\mathscr{H}_1(A)|-1)$-tuples $(x_{i_B})_{B\in\mathscr{H}_1(A),B\not\ni 2}$ satisfying (i)–(iv) of Definition 6.1 as well as (7.5). For $A \supsetneq C$ we define $J_A(x_1, x_2, r)$ from (6.2), with J_B replaced by J'_B for $B\in\mathscr{H}_0(A)$ and with integration region Z'_A given by Z_A with the additional restriction $r_A \geqslant |x_1-x_2|$.

Now Lemma 6.2 clearly holds for $|A\cap I_0|\leqslant 1$. Then as in (6.5),

$$\begin{aligned} J'_C &\leqslant Cd^{\Sigma_C\tilde{\mu}_B}[1+\log(r/d)]^{\Sigma_C m_B}\int_{b_1|x_1-x_2|}^{b_2|x_1-x_2|} r_C^{\mu_C-\Sigma_C\tilde{\mu}_B-2}\frac{dr_C}{r_C} \\ &\leqslant Cd^{\Sigma_C\tilde{\mu}_B}[1+\log(r/d)]^{m_C}|x_1-x_2|^{\mu_C-\Sigma_C\tilde{\mu}_B-2} \end{aligned}$$

By a further induction, for $A\supset C$,

$$\begin{aligned} J'_A &\leqslant Cd^{\Sigma_C\tilde{\mu}_B+\Sigma'_A\tilde{\mu}_B}|x_1-x_2|^{\tilde{\mu}'_D-\Sigma_C\tilde{\mu}_B-2}r^{\mu_A-\Sigma_A\tilde{\mu}'_B} \\ &\quad\times[1+\log(r/d)]^{m_C}[1+\log(r/|x_1-x_2|)]^m \end{aligned} \tag{7.6}$$

for some $m>0$, where $D\in\mathscr{H}_0(A)$ satisfies $C\subset D$, $\sum'_A$ is defined as $\sum_{\{B\in\mathscr{H}_0(A)|B\neq D\}}$, and $\tilde{\mu}'_B$ defined as in (6.1) but with the partition $\mathscr{D}$ restricted as in (7.2). Finally we estimate $\rho^T_{\mathbf{n},\mathbf{N}}$ as a sum (over $\mathscr{H}$ and $\mathscr{P}$) of terms $J_I(x_1, x_2, \infty)$; the corresponding integral is convergent by (7.2) and satisfies

$$\int_{|x_1-x_2|}^{\infty} r_I^{\mu_I-\Sigma_I\tilde{\mu}'_B}[1+\log(r/|x_1-x_2|)]^m\frac{dr_I}{r_I}\leqslant C|x_1-x_2|^{\mu_I-\Sigma_I\tilde{\mu}'_B} \tag{7.7}$$

Combining (7.6) and (7.7) yields (7.3); the resulting asymptotic power bound $\mu_I-\sum'_I\tilde{\mu}'_B-\sum_C\tilde{\mu}_B-2$ is shown to be bounded by $-\zeta$ as in the proof of (6.6). ∎

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